Supergravity p-branes revisited: extra parameters, uniqueness, and topological censorship

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We perform a complete integration of the Einstein-dilaton-antisymmetric form action describing black p-branes in arbitrary dimensions assuming the transverse space to be homogeneous and possessing spherical, toroidal or hyperbolic topology. The generic solution contains eight parameters satisfying one constraint. Asymptotically flat solutions form a five-parametric subspace, while conditions of regularity of the non-degenerate event horizon further restrict this number to three, which can be related to the mass and the charge densities and the asymptotic value of the dilaton. In the case of a degenerate horizon, this number is reduced by one. Our derivation constitutes a constructive proof of the uniqueness theorem for p-branes with the homogeneous transverse space. No asymptotically flat solutions with toroidal or hyperbolic transverse space within the considered class are shown to exist, which result can be viewed as a demonstration of the topological censorship for p-branes. From our considerations it follows, in particular, that some previously discussed p-brane-like solutions with extra parameters do not satisfy the standard conditions of asymptotic flatness and absence of naked singularities. We also explore the same system in presence of a cosmological constant, and derive a complete analytic solution for higher-dimensional charged topological black holes, thus proving their uniqueness.

I. INTRODUCTION

Classical solutions of the supergravity equations, describing p-branes charged with respect to a $p + 1$ antisymmetric form field, were extensively studied during the past decade [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. In the case of a single brane (which will be the only one discussed here), the standard (black) brane solution depends on two parameters, the mass and the charge (densities) of the brane [1, 3], these solutions are asymptotically flat and possess a regular event horizon. Black p-branes have the $ISO(p) \times R$ symmetry of the world volume (with R corresponding to the time direction), which is enhanced to the full Poincaré symmetry $ISO(p, 1)$ in the extremal (BPS) case. In the simplest case the space transverse to the brane is taken to be spherically symmetric, the generalizations to the products of the lower-dimensional sphere by the flat space are also known.

Both the BPS and the black branes were first obtained by solving the corresponding field equations under special ansätze for the metric apart from the above symmetries [3, 7], so the degree of generality of these solutions was not clear a priori. Alternatively, they may be generated from a suitably smeared Schwarzschild solution by Harrison-type transformations [11]. The question of the uniqueness of the standard p-brane solutions was raised recently [13], but the explicit proof was given only in the case $p = 0$, that is for multidimensional black holes, and only for a non-degenerate event horizon. Meanwhile, some more general solutions to the same system depending on higher number of parameters were also suggested [6, 12] for $ISO(p, 1)$ symmetric branes. More recently a complete integration of the Einstein-dilaton-antisymmetric form system for a single brane was performed [14] and a family of $ISO(p) \times R$ solutions was presented, containing four free parameters. An interpretation of one of the extra parameters was attempted in [17] (see also [18]): the $ISO(p, 1)$ subfamily of the solutions of [14] was treated as describing a brane-antibrane system in the sense of Sen [15, 16], the corresponding extra degree of freedom being associated with the tachyon. The solutions of [14] were also invoked in some recent attempts to find a supergravity description for stable non-BPS branes in string theory [20, 21, 22, 23]. Without entering into

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a discussion of the consistency of these suggestions, we would like here to stress that the detailed structure of singularities of the p-brane-like solutions with extra parameters was not investigated so far. Other generalizations of the $ISO(p,1)$ solution were given in [19]. Besides containing additional parameters, the solutions presented in this paper also describe a more general structure of the transverse space, namely, $SO(k) \times R^q$, $q = D - p - k - 2$ (cylinder), R^{q+k+1}, as well as the case of the hyperbolic geometry $SO(k-1,1) \times R^q$, . The latter two cases were previously explored for $p = 0$ (topological black holes) in the presence of a negative cosmological constant, causing the space-time to possess an asymptotic AdS structure [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. It can be assumed that the flat and the hyperbolic spaces are compactified by factoring the space by a discrete subgroup of the isometry group [29] and describe black holes with toroidal and higher genus surfaces of the event horizon (for a recent review see [35]). Whether such solutions could be extended to p-branes remains unclear, since no analytic solutions for p-branes with cosmological constant were found so far [36]. Neither are we aware of any topological censorship arguments for p-branes of the type known for black holes [37, 38].

The aim of this paper is to clarify the nature of generic solutions to Einstein-dilaton-antisymmetric form equations describing p-branes with spherical, toroidal and hyperbolic geometry of the transverse space. We construct generic solutions by reducing the system to two Liouville equations, and then carefully investigate restrictions on the free parameters following from the requirements of the asymptotic flatness and the absence of naked singularities. We find that in both cases of non-degenerate, or degenerate event horizons, the generic solution is reduced effectively to the standard black brane or the BPS solution, fully characterized by the tension, the charge density and the asymptotic value of the dilaton (this parameter was often ignored in the previous literature). No extra parameters can be added without violating the regularity conditions. This proves the uniqueness theorem for

p-branes for both non-degenerate and degenerate event horizons, in the case of $ISO(p) \times R$ and $ISO(p,1)$ symmetries of the world-volume and a spherical transverse space. We also prove the absence of asymptotically flat p-brane solutions with toroidal and hyperbolic transverse geometries, thus generalizing the topological censorship conjecture for p-branes. We also explore the same system in the presence of a cosmological constant, in the hope that topological p-brane solutions will arise. But in this case the system is not fully integrable, and we were only able to give a simple constructive derivation of the topological 0-brane with AdS asymptotics.

II. GENERAL SETTING

We consider the action containing a graviton, a q-form field strength $F_{[q]}$, and a dilaton scalar ϕ coupled to the form field with coupling constant a. This is a general framework which encompasses the bosonic sector of various bosonic models, coming from a truncation of the low energy limit of M-theory and string theories, for a certain choice of the dimension D, the rank k of the form field, and the dilaton coupling a. In the Einstein frame, the action is given by

$$ S = \int d^D x \sqrt{-g} \left(R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2\,k!} e^{a\phi} F_{[k]}^2 \right). \quad (1) $$

The corresponding field equations are invariant under the following discrete S-duality:

$$ g_{\mu\nu} \to g_{\mu\nu}, \qquad F \to e^{-a\phi} * F, \qquad \phi \to -\phi, \quad (2) $$

where $*$ denotes the d-dimensional Hodge dual. This may be used to construct electric versions of magnetic S-branes and vice versa, so here we will consider only magnetic solutions. The equations of motion, derived from the variation of the action with respect to the individual fields, are

$$ R_{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{e^{a\phi}}{2(k-1)!} \left[F_{\mu\alpha_2\cdots\alpha_k} F_\nu{}^{\alpha_2\cdots\alpha_k} - \frac{k-1}{k(D-2)} F_{[k]}^2 \, g_{\mu\nu} \right] = 0, \quad (3) $$

$$ \partial_\mu \left(\sqrt{-g} \, e^{a\phi} \, F^{\mu\nu_2\cdots\nu_k} \right) = 0, \quad (4) $$

$$ \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \partial^\mu \phi \right) - \frac{a}{2\,k!} e^{a\phi} F_{[k]}^2 = 0. \quad (5) $$

We study p-branes with a world volume given by a $p+1$ dimensional space with isometries $ISO(p) \times R$ and with a transverse space being the $\tilde{d} + 1$ dimensional space $\Sigma_{k,\sigma}$. We will use the standard notation for dimensions $d =$

$p+1$, $\tilde{d} = k - 1 = D - d - 2$. The space-time interval

$$ ds^2 = -e^{2B} dt^2 + e^{2D}(dx_1^2 + \cdots + dx_p^2) + e^{2C} d\Sigma_{k,\sigma}^2 + e^{2A} dr^2, \quad (6) $$

is parameterized by four functions $A(r)$, $B(r)$, $C(r)$ and $D(r)$ depending only on r. The transverse space $\Sigma_{k,\sigma}$ for

$\sigma = 0, +1, -1$ is the k-dimensional flat space, the sphere and the hyperbolic space respectively:

$$d\Sigma_{k\sigma}^2 = \bar{g}_{ab}dy^a dy^b = \begin{cases} d\psi^2 + \sinh^2\psi\, d\Omega_{\tilde{d}}^2, & \sigma = -1, \\ d\psi^2 + \psi^2\, d\Omega_{\tilde{d}}^2, & \sigma = 0, \\ d\psi^2 + \sin^2\psi\, d\Omega_{\tilde{d}}^2, & \sigma = 1, \end{cases}$$
(7)

The Ricci tensor for the transverse space reads

$$\bar{R}_{ab} = \sigma \tilde{d} \bar{g}_{ab}.$$
(8)

The metrics (7) have $SO(\tilde{d}, 1)$, $ISO(k)$ and $SO(k)$ isometries respectively. In the flat and hyperbolic case one can assume suitable compactifications by factoring over an appropriate discrete subgroup of the isometry group, e.g. for $\sigma = 0$ one can choose a torus, for $\sigma = -1$ – some compact hyperbolic space [39].

With this ansatz, the equation for the form field (4), can be easily be solved,

$$F_{[k]} = b\, \text{vol}(\Sigma_{k,\sigma}),$$
(9)

where b is the field strength parameter, and $\text{vol}(\Sigma_{k,\sigma})$ denotes the volume form of the space $\Sigma_{k,\sigma}$.

The Ricci tensor for the metric (6) has the non-vanishing components

$$R_{tt} = e^{2B-2A}\left(B'' + B'F'\right),$$
(10)
$$R_{xx} = -e^{2D-2A}\left(D'' + D'F'\right),$$
(11)
$$R_{rr} = -F'' - A'' - B'(B'-A') - (\tilde{d}+1)C'(C'-A') - (d-1)D'(D'-A'),$$
(12)
$$R_{ab} = \left(-e^{2C-2A}\left(C'' + C'F'\right) + \sigma\tilde{d}\right)\bar{g}_{ab},$$
(13)

where

$$F = B - A + (\tilde{d}+1)C + (d-1)D.$$
(14)

Using the expressions for the Ricci tensor and substituting the form field (9), we find three equations for B, C and D with similar differential operators

$$B'' + B'F' = \frac{\tilde{d}b^2 e^{G-2F}}{2(D-2)},$$
(15)

$$C'' + C'F' = -\frac{db^2 e^{G-2F}}{2(D-2)} + \sigma\tilde{d}e^{2(A-C)},$$
(16)

$$D'' + D'F' = \frac{\tilde{d}b^2 e^{G-2F}}{2(D-2)},$$
(17)

where

$$G = a\phi + 2B + 2(d-1)D,$$
(18)

and the following equation involving the function A:

$$(A+F)'' - A'(A+F)' + (\tilde{d}+1)C'^2 + (d-1)D'^2 + \frac{1}{2}\phi'^2 = \frac{\tilde{d}b^2 e^G}{2(D-2)}.$$
(19)

The dilaton equation Eq.(5) takes the following form

$$\phi'' + \phi'F' = \frac{ab^2 e^{G-2F}}{2},$$
(20)

where primes denote derivatives with respect to r.

To simplify the above system we introduce a new independent variable via

$$d\tau = \tilde{d}\, e^{-F} dr,$$
(21)

and pass to a new function

$$\mathcal{A} = A + F.$$
(22)

Then, denoting the derivatives with respect to τ by dot, we obtain the following system:

$$\ddot{B} = \frac{b^2 e^G}{2\tilde{d}(D-2)},$$
(23)
$$\ddot{D} = \frac{b^2 e^G}{2\tilde{d}(D-2)},$$
(24)
$$\ddot{\phi} = \frac{ab^2 e^G}{2\tilde{d}^2},$$
(25)
$$\ddot{C} = -\frac{b^2 d e^G}{2\tilde{d}^2(D-2)} + \frac{\sigma}{\tilde{d}}e^{2(\mathcal{A}-C)},$$
(26)

and

$$\ddot{\mathcal{A}} - \dot{\mathcal{A}}^2 + \dot{B}^2 + (\tilde{d}+1)\dot{C}^2 + (d-1)\dot{D}^2 + \frac{1}{2}\dot{\phi}^2 = \frac{b^2 e^G}{2\tilde{d}(D-2)}.$$
(27)

Note that the function F has disappeared from the equations, showing F to be a gauge function. Under reparameterization of the radial coordinate it can be chosen arbitrarily, in particular, set to zero. Once the system (23-26) is solved, the function $\mathcal{A}$ can be expressed through B, C, D, and the Eq. (27) becomes a constraint equation. Therefore the complete solution for the metric functions and the dilaton should contain eight free parameters subject to one constraint. One of them, however, is redundant, since the the system is autonomous. In return, we have already introduced one free parameter b related to the form field strength (9), so the actual number of free parameters that we expect in the complete solution has to be seven.

The Ricci scalar calculated for the solutions to the field equations can be written in the following form

$$R = \frac{1}{2}\phi'^2\, e^{-2A} + \frac{(d-\tilde{d})b^2}{2(D-2)}e^{a\phi - 2(\tilde{d}+1)C},$$
(28)

which is manifestly invariant under reparameterizations of the radial coordinate.

III. THE COMPLETE SOLUTION

The above system can be integrated as follows. First we observe that the functions B, D and $\tilde{d}\phi/(a(D-2))$

may differ only by a solution of the homogeneous equation, which is a linear function of τ, thus we obtain D and ϕ in terms of B as follows:

$$D = B + d_1\tau + d_0, \tag{29}$$

$$\phi = \frac{a(D-2)}{\tilde{d}}B + \phi_1\tau + \phi_0, \tag{30}$$

where d_0, d_1, ϕ_0, ϕ_1 are free constant parameters. Substituting this into the Eq. (18) one finds the following relation:

$$G = \frac{\Delta(D-2)}{\tilde{d}}B + g_1\tau + g_0, \tag{31}$$

where

$$\Delta = a^2 + \frac{2d\tilde{d}}{D-2}, \tag{32}$$

and the integration constants combine to

$$g_{0,1} = a\phi_{0,1} + 2(d-1)d_{0,1}. \tag{33}$$

Together with the Eq. (23) one then obtains a decoupled Liouville equation for G:

$$\ddot{G} = \frac{b^2\Delta}{2\tilde{d}^2}\,\mathrm{e}^G, \tag{34}$$

from which the following first integral is found straightforwardly:

$$\dot{G}^2 - \frac{b^2\Delta}{\tilde{d}^2}\,\mathrm{e}^G = \alpha^2, \tag{35}$$

with a new integration constant α which can be real or pure imaginary (for definiteness we will assume real α to be non-negative). Note that under rescaling $\tau \to k^{-1}\tau$ the parameter α will scale as $\alpha \to k\alpha$. The general solution of (34) for real $\alpha \neq 0$ reads:

$$G = \ln\left(\frac{\alpha^2\tilde{d}^2}{\Delta b^2\,\sinh^2\left(\frac{\alpha}{2}(\tau-\tau_0)\right)}\right), \tag{36}$$

where τ_0 is another integration constant. For $\alpha = 0$ one has instead

$$G = \ln\left(\frac{4\tilde{d}^2}{\Delta b^2(\tau-\tau_0)^2}\right). \tag{37}$$

For imaginary $\alpha = i\bar{\alpha}$ the solution takes the form

$$G = \ln\left(\frac{\bar{\alpha}^2\tilde{d}^2}{\Delta b^2\,\sin^2\left(\frac{\bar{\alpha}}{2}(\tau-\tau_0)\right)}\right), \tag{38}$$

Combining now the remaining equations, one can show that the linear combination

$$H = 2(\mathcal{A} - \mathcal{C}) \tag{39}$$

obeys a second decoupled Liouville equation

$$\ddot{H} = 2\sigma\mathrm{e}^H, \tag{40}$$

admitting the first integral

$$\dot{H}^2 - 4\sigma\mathrm{e}^H = \beta^2. \tag{41}$$

For $\sigma = 1$ this parameter can be real (in which case it will be assumed non-negative) or pure imaginary, while for $\sigma = 0, -1$ a solution exists only for β real. For real positive β we find the following solutions for all values of σ:

$$H = \left\{\begin{array}{ll} 2\ln\beta/2 - \ln\left[\sinh^2(\beta\tau/2)\right], & \sigma = 1, \\ \pm\beta\tau, & \sigma = 0, \\ 2\ln\beta/2 - \ln\left[\cosh^2(\beta\tau/2)\right], & \sigma = -1. \end{array}\right\} \tag{42}$$

Note that we could introduce here another integration constant replacing τ by $\tau - \tau_1$ as in (36), but since the initial system is autonomous, one can choose without loss of generality $\tau_1 = 0$ (while keeping τ_0 in the solution for G). For $\beta = 0$ one has, for $\sigma = 1$:

$$H = -\ln\tau^2, \tag{43}$$

and for $\sigma = 0$

$$H = H_0 \tag{44}$$

constant, while for pure imaginary $\beta = i\bar{\beta}$ and $\sigma = 1$

$$H = \ln\left(\frac{\bar{\beta}^2}{4\sin^2\left(\bar{\beta}\tau/2\right)}\right). \tag{45}$$

Finally, expressing the metric functions A, C from (14), (39), one can write the full solution in terms of G, H as follows:

$$B = \frac{\tilde{d}}{\Delta(D-2)}\left(G - g_1\tau - g_0\right), \tag{46}$$

$$D = \frac{\tilde{d}}{\Delta(D-2)}\left(G - g_1\tau - g_0\right) + d_1\tau + d_0, \tag{47}$$

$$C = \frac{1}{2\tilde{d}}H - \frac{d}{\Delta(D-2)}G + c_1\tau + c_0, \tag{48}$$

$$A = \frac{(1+\tilde{d})}{2\tilde{d}}H - \frac{d}{\Delta(D-2)}G - F + c_1\tau + c_0, \tag{49}$$

$$\phi = \frac{a}{\Delta}G + f_1\tau + f_0, \tag{50}$$

where

$$c_{0,1} = \frac{a}{\Delta}\left(\frac{d}{D-2}\phi_{0,1} - \frac{(d-1)a}{\tilde{d}}d_{0,1}\right),$$

$$f_{0,1} = \phi_{0,1} - \frac{a}{\Delta}g_{0,1} = \frac{2\tilde{d}}{a}c_{0,1}. \tag{51}$$

The gauge function F remains arbitrary and can be used to fix the gauge in any convenient form. Our

complete solution thus depends on eight parameters: $b, d_0, d_1, \phi_0, \phi_1, \tau_0, \alpha, \beta$, from which the following four $d_1, \phi_1, \alpha, \beta$ are subject to a constraint resulting from the Eq. (27):

$$\frac{(\tilde{d}+1)\beta^2}{4\tilde{d}} - \frac{\alpha^2}{2\Delta} - \tilde{d}c_1^2 - (d-1)\left(\frac{dg_1}{\Delta(D-2)} - d_1\right)^2 +$$

$$+ \left(\frac{\tilde{d}g_1}{\Delta(D-2)}\right)^2 - \frac{f_1^2}{2} = 0. \quad (52)$$

Therefore, we have seven free parameters in the generic solution. From this equation it follows that β can be pure imaginary only if α is also imaginary, while $\beta = 0$ is possible only for α imaginary or $\alpha = 0$ (the other parameters d_1 and ϕ_1 also vanishing).

IV. ASYMPTOTIC FLATNESS

A. Spherical transverse space

Since the complete solution is determined by two functions $G(\tau)$, $H(\tau)$ and a set of linear functions of τ, one has to investigate the behavior of the metric when $\tau \to 0$, $\tau \to \tau_0$, and $\tau \to \pm\infty$. It is easy to see, that for $\sigma = 1$, in the limit $\tau \to 0$ we find a Minkowski space after suitably choosing some free parameters. The function H in this limit is

$$H \approx \ln\frac{1}{\tau^2}, \quad (53)$$

while G tends to a finite value

$$G_0 = \ln\left(\frac{\alpha^2\tilde{d}^2}{\Delta b^2 \sinh^2(\alpha\tau_0/2)}\right), \quad (54)$$

if $\tau_0 \neq 0$. Imposing the following two conditions on the parameters

$$d_0 = 0, \quad (55)$$
$$a\phi_0 = G_0, \quad (56)$$

one obtains

$$B \approx 0, \quad (57)$$
$$D \approx 0, \quad (58)$$
$$C \approx \ln\left(\frac{1}{|\tau|}\right)^{1/\tilde{d}}, \quad (59)$$
$$A \approx \ln\left(\frac{1}{|\tau|}\right)^{(1+1/\tilde{d})}. \quad (60)$$

Choosing the gauge function to behave asymptotically as

$$F_\infty = \ln r^{\tilde{d}+1}, \quad (61)$$

one obtains from the Eq. (21)

$$\tau = -r^{-\tilde{d}}, \quad (62)$$

so transforming to the radial variable r we find the interval in the vicinity of the section $\tau = 0$ ($r \to \infty$) as

$$ds^2 = -dt^2 + dx_1^2 + \cdots + dx_p^2 + r^2 d\Sigma_{\tilde{d}+1,1}^2 + dr^2. \quad (63)$$

The asymptotic value of the dilaton is finite

$$\phi_\infty = \phi_0. \quad (64)$$

The case $\tau_0 = 0$, which leads to non-asymptotically flat solutions, shall be examined elsewhere [40].

B. Flat or hyperbolic transverse space: topological censorship

For $\sigma = 0$ the quantity H is a linear function of τ, so when $\tau \to 0$ all the metric functions go to constant values if $\tau_0 \neq 0$. Therefore the space-time is locally cylindrical,

$$ds^2 = -dt^2 + dx_1^2 + \cdots + dx_p^2 + r_0^2 d\Sigma_{\tilde{d}+1,0}^2 + d\tau^2, \quad (65)$$

(after a suitable rescaling of the world-volume coordinates) with r_0 constant, so that the regular timelike section $\tau = 0$ is at finite distance. In the hyperbolic case $\sigma = -1$

$$H \to \ln\frac{\beta^2}{4} \quad (66)$$

as $\tau \to 0$, so the situation is similar. For $\tau \to \tau_0 \neq 0$, $G \to +\infty$, so that the section $\tau = \tau_0$ is singular and again at finite distance.

Finally we inquire whether an asymptotic region, say $\tau = +\infty$, can be a regular region at spatial infinity? When $\tau \to +\infty$, $G \sim -\alpha\tau$. For asymptotic flatness we require $B \to 0$ and $D \to 0$, leading to

$$g_1 = -\alpha, \quad d_1 = 0. \quad (67)$$

Then,

$$c_1 = -\frac{\alpha d}{\Delta(D-2)}, \quad (68)$$

so that

$$\mathcal{A} \sim \frac{1+\tilde{d}}{2\tilde{d}}H \quad (69)$$

goes to $+\infty$ only for $\sigma = 0$ ($H \sim \beta\tau$). Accordingly the metric asymptotes to

$$ds^2 \sim -dt^2 + dx_1^2 + \cdots + dx_p^2 + r^{\frac{2}{1+\tilde{d}}} d\Sigma_{\tilde{d}+1,0}^2 + dr^2 \quad (70)$$

($r \sim e^{\mathcal{A}} \sim e^{\frac{1+\tilde{d}}{2\tilde{d}}\beta\tau}$). This is asymptotically flat only in the trivial case $\tilde{d} = 0$.

We conclude that the asymptotic flatness requirement cannot be fulfilled for flat (toroidal) or hyperbolic transverse spaces, so there are no asymptotically flat "topological" branes. In the particular case of black holes ($d = 1$) this is a topological censorship theorem which was proved recently [37, 38] for arbitrary space-time dimension with less assumptions than here. Our argument shows that this is likely to be extendible to the brane case as well.

V. HORIZONS

From now on we will consider only spherical transverse space. Horizons may arise in the asymptotic regions $\tau \to \pm\infty$, when the metric function e^{2B} vanishes. In these regions the function G behaves (for $\alpha > 0$) as

$$G \approx -\alpha|\tau| + \text{const},\tag{71}$$

so depending on the ratio between parameters, one can get horizons in both regions. Since the asymptotic region is at $\tau = 0$ we will assume that the outer horizon (the event horizon), if any, is located at $\tau = -\infty$, and an internal Cauchy horizon (if any) – at $\tau = +\infty$.

In view of (71), in the event horizon region one has

$$e^{2B} \approx \text{const} \cdot \exp\left(\frac{2\tilde{d}}{\Delta(D-2)}(\alpha - g_1)\tau\right).\tag{72}$$

This goes to zero if

$$\alpha > g_1.\tag{73}$$

Now we have to distinguish the cases of non-degenerate and degenerate horizons. Considering the prolongation of geodesics through the horizon, we finds that in terms of the affine parameter λ, related to τ via

$$d\lambda = e^{(\mathcal{A}+B)}d\tau,\tag{74}$$

one has to demand

$$e^{2B} \sim \lambda^n\tag{75}$$

on the horizon $(\lambda = 0)$ where $n = 1$ in the non-degenerate and $n \geq 2$ in the degenerate case.

A. Non-degenerate horizon

In the case $n = 1$ from the above reasoning we find the following condition

$$e^{-(B+\mathcal{A})}\frac{d}{d\tau}e^{2B} \to \text{const}, \text{ as } \tau \to -\infty.\tag{76}$$

In view of the Eqs. (72), (73) it is clear that $\dot{B}$ is non-zero and

$$e^{\mathcal{A}} \sim e^{B},\tag{77}$$

so that the exponential $e^{\mathcal{A}}$ also vanishes on the horizon. We then find that the Ricci scalar (28) diverges unless $\dot\phi$ vanishes in the limit $\tau \to -\infty$. Demanding this quantity to vanish we obtain the following condition on the parameters:

$$c_1 = -\frac{\alpha a^2}{2\tilde{d}\,\Delta}.\tag{78}$$

Now rewrite the constraint equation with account for the definition of H as follows

$$-\dot{\mathcal{A}}^2 + \dot{B}^2 + (\tilde{d}+1)\dot{C}^2 + (d-1)\dot{D}^2 + \frac{1}{2}\dot\phi^2$$
$$= \frac{b^2}{2\tilde{d}^2}e^G - \frac{\tilde{d}+1}{\tilde{d}}e^H.\tag{79}$$

It is easy to see that both quantities on the right hand side of this equation vanish on the horizon, while the first two terms on the left hand side mutually cancel. From the positivity of the remaining part one finds that $\dot{C}$ and $\dot{D}$ should also vanish on the horizon. If we impose both these conditions then the constraint equation on the parameters (52) will be automatically satisfied. Together with (78) this gives the following relations

$$\alpha = \beta = -2d_1 = -\frac{2\tilde{d}}{a(D-2)}\phi_1.\tag{80}$$

Thus the regularity of the event horizon fixes two more parameters (the third follows from the constraint). It is then easy to check that if $\tau_0 > 0$ our solution does not possess any singularity on the semi-axis $-\infty < \tau < 0$, so solutions are free from naked singularities.

Of course, the behavior inside the horizon can be expected to be singular, and one can show that the point $\tau = \tau_0$ is just such a singularity. Substituting the solution obtained in the vicinity of $\tau = \tau_0$ into the Eq. (28) one obtains

$$R \sim \left(\frac{1}{\tau - \tau_0}\right)^{2\left(1 + \frac{2d}{\Delta(D-2)}\right)}.\tag{81}$$

It is convenient to choose the Schwarzschild gauge which allows to parameterize the entire space-time in a more transparent form. The corresponding gauge function F compatible with the asymptotic choice (61) looks as follows

$$F = \ln\left(r^{\tilde{d}+1}f_-f_+\right),\tag{82}$$

with

$$f_\pm = 1 - \frac{x_\pm}{x}, \quad x = r^{\tilde{d}}, \quad x_\pm = r_\pm^{\tilde{d}}, \quad x_- < x_+,\tag{83}$$

where the positive values $r_\pm$ correspond to the location of horizons in accord with previous assumptions. Indeed, integrating the equation (21) for τ we obtain

$$\tau = \frac{1}{x_+ - x_-}\ln\frac{f_+}{f_-}.\tag{84}$$

By fixing the gauge in the above way we have introduced two extra parameters $r_\pm$, so now we can fix the scale, trading α for the difference $x_+ - x_-$:

$$\alpha = x_+ - x_-.\tag{85}$$

In this case the function H will take the simple form for $x > x_+$ and $x < x_-$:

$$H = \ln(x - x_+)(x - x_-). \qquad (86)$$

Note that the function $\tau(x)$ becomes complex in the region $x_- < x < x_+$ between the horizons, but this does not make the solution in terms of x complex. Passing through the horizon the variable τ shifts into the complex plane by $i\pi/\alpha$, which is precisely what is needed to account for the necessary sign change of the exponential e^G, positive for real τ, and negative for $\tau = \mathrm{Re}(\tau) + i\pi/\alpha$. Thus the outer region $r > r_+$ maps to the half of the real axis $-\infty < \tau < 0$, the interval $r_- < r < r_+$ maps to the line in the complex plane of τ parallel to the real axis and shifted by $i\pi/\alpha$, with the point r_- corresponding to $\mathrm{Re}(\tau) = \infty$, while the region $r < r_-$ (if the section $r = r_-$ is non-singular, which is possible only in the absence of the dilaton) maps to the part of the positive real axis of τ from infinity to $\tau = \tau_0$. Let $x_0 = r_0^{\tilde{d}}$ be the image of τ_0 (we assume that $r_0 < r_-$). In terms of these quantities the function G can be presented as follows

$$e^G = \frac{4\tilde{d}^2(x_+ - x_0)(x_- - x_0)}{\Delta b^2} f_+ f_- f_0^{-2}, \qquad (87)$$

where

$$f_0 = 1 - \frac{x_0}{x}. \qquad (88)$$

With this parametrization the solution now takes the following form

$$ds^2 = \left(\frac{f_-}{f_0}\right)^{\frac{4\tilde{d}}{\Delta(D-2)}} \left(-\frac{f_+}{f_-} dt^2 + d\mathbf{x}^2\right) + $$
$$+ f_-^{\frac{2a^2}{\Delta\tilde{d}}} f_0^{\frac{4d}{\Delta(D-2)}} \left(r^2 d\Sigma_{k,1} + \frac{dr^2}{f_+ f_-}\right), \quad (89)$$

$$e^{\phi - \phi_0} = \left(\frac{f_-}{f_0}\right)^{\frac{2a}{\Delta}}. \qquad (90)$$

Here ϕ_0 is given by Eq. (56) as a function of other parameters b, τ_0, so totally the solution seems to depend on four free parameters b, $r_\pm$, r_0. This differs from the standard black brane solution (with an arbitrary value of the dilaton at infinity, usually assumed zero) only by the presence of the factors f_0. According to the Eq. (81) the point $r = r_0$ is a curvature singularity, while, as it easy to check, the point $r = 0$ is not. So with our choice of coordinates the central singularity is at $r = r_0$, and by the coordinate transformation $x \to x + x_0$ or

$$r \to \left(r^{\tilde{d}} + r_0^{\tilde{d}}\right)^{1/\tilde{d}}, \qquad (91)$$

it can be moved to $r = 0$. This amount to set $x_0 = 0$ in (89), and we arrive at the standard form of the black brane solution [3]

$$ds^2 = f_-^{\frac{4\tilde{d}}{\Delta(D-2)}} \left(-\frac{f_+}{f_-} dt^2 + d\mathbf{x}^2\right) + $$

$$+ f_-^{\frac{2a^2}{\Delta\tilde{d}}} \left(r^2 d\Sigma_{k,1} + \frac{dr^2}{f_+ f_-}\right), \qquad (92)$$

$$e^{\phi - \phi_0} = f_-^{\frac{2a}{\Delta}}, \qquad (93)$$

with only three free parameters.

B. Degenerate horizon

For a double horizon ($n = 2$ in (75)), we have near the horizon

$$e^{2B} \sim \lambda^2, \quad \frac{de^{2B}}{d\lambda} = e^{B-\mathcal{A}} \dot{B} \sim \lambda, \qquad (94)$$

leading to

$$e^{-\mathcal{A}} \dot{B} \sim 1. \qquad (95)$$

A first possibility is that both $\dot{B}$ and $e^{\mathcal{A}}$ are finite on the horizon, implying $\dot{\mathcal{A}} \sim 0$. However the constraint equation (79) then implies, in particular, that $\dot{B} = 0$, contrary to the assumption. So the solution of (95) is that both $\dot{B}$ and $e^{\mathcal{A}}$ vanish on the horizon. The first condition implies

$$\alpha = g_1, \qquad (96)$$

then the condition $e^B = 0$ can only be satisfied if $\alpha = 0$, in which case the solution for G is given by (37). We then see from the second condition $e^{\mathcal{A}} \sim 0$ that the Ricci scalar (28) diverges on the horizon unless $\dot{\phi} \sim 0$. This condition fixes the value of the constant $f_1 = 0$. Together with the previous condition $g_1 = 0$ this gives

$$d_1 = \phi_1 = 0. \qquad (97)$$

Now from the constraint equation (52) we see that $\beta = 0$ so the solution for H must be taken in the form (43) (and $\sigma = 1$). Finally, we obtain the conditions

$$\alpha = \beta = d_1 = \phi_1 = 0, \qquad (98)$$

which, together with our choice of a scale (85) naturally fit the previous conditions (80) in the limit

$$r_- \to r_+. \qquad (99)$$

VI. COSMOLOGICAL CONSTANT

Here we investigate the same system in presence of the cosmological constant. For simplicity we set the dilaton to zero since already the black hole case leaves a little hope to find analytic solutions in this case. The system of equations takes the following form (in r-terms):

$$B'' + B'F' = \frac{\tilde{d}b^2}{2(D-2)}\, \mathrm{e}^{2(A-(\tilde{d}+1)C)} + \Lambda \mathrm{e}^{2A}, \tag{100}$$

$$C'' + C'F' = -\frac{db^2}{2(D-2)}\, \mathrm{e}^{2(A-(\tilde{d}+1)C)} + \sigma\tilde{d}\mathrm{e}^{2(A-C)} + \Lambda \mathrm{e}^{2A}, \tag{101}$$

$$D'' + D'F' = \frac{db^2}{2(D-2)}\, \mathrm{e}^{2(A-(\tilde{d}+1)C)} + \Lambda \mathrm{e}^{2A}, \tag{102}$$

$$B'' + B'(B'-A') + (\tilde{d}+1)\left[C'' + C'(C'-A')\right] + (d-1)\left[D'' + D'(D'-A')\right] =$$
$$= \frac{\tilde{d}b^2}{2(D-2)}\, \mathrm{e}^{2(A-(\tilde{d}+1)C)} + \Lambda \mathrm{e}^{2A}, \tag{103}$$

where no specific gauge was imposed.

Note that our ansatz for $\sigma = 0, -1$ corresponds to 'topological' solutions, so our aim was to explore whether topological black holes [35] admit p-brane generalizations. The main difference in the structure of the systems (100–103) and the system considered in Sec. 2 is that now one has three different exponentials instead of two for zero cosmological constant. So, with some luck, one could hope to obtain three Liouville equations. We could proceed along the same lines choosing e.g. the gauge $F = 0$, but this does not lead to decoupling in the general case: we were not able to reduce the resulting system to a set of Liouville equations or to integrate it in any other way except for the case $d = 1$ corresponding to black holes. So to answer the question about existence

of topological branes numerical work is likely to be required. This is beyond the scope of the present paper, so we will proceed in obtaining a complete solution for the case $d = 1$ only. We obtain the solution which was known earlier, but contrary to previous treatment we find it via a complete integration of the corresponding system of equations, thus providing the uniqueness proof for topological zero-branes with homogeneous transverse space.

For $d = 1$ the D-part of the equations disappears, and assuming the curvature gauge

$$C = \ln r, \tag{104}$$

we are left with the system

$$B'' + B'(B-A') + \frac{\tilde{d}+1}{r}\, B' = \left(\frac{\tilde{d}b^2}{2(D-2)r^{2(\tilde{d}+1)}} + \Lambda\right)\mathrm{e}^{2A}, \tag{105}$$

$$\frac{\tilde{d}}{r^2} + \frac{1}{r}\,(B'-A') = \left(\sigma\tilde{d} - \frac{b^2}{2(D-2)r^{2\tilde{d}}} + \Lambda\right)\mathrm{e}^{2A}, \tag{106}$$

$$B'' + B'(B'-A') - \frac{\tilde{d}+1}{r}\, A' = \left(\frac{\tilde{d}b^2}{2(D-2)r^{2(\tilde{d}+1)}} + \Lambda\right)\mathrm{e}^{2A}, \tag{107}$$

Comparing the Eqs. (105) and (107) we see that

$$B = -A + \text{const}, \tag{108}$$

where the integration constant can be removed by rescaling of time. Then the Eq. (106) becomes a first order decoupled equation for A which can be rewritten as

$$r^{1-\tilde{d}}\frac{d}{dr}\left(r^{\tilde{d}}\mathrm{e}^{-2A}\right) = \sigma\tilde{d} - \frac{b^2}{2(D-2)r^{2\tilde{d}}} + r^2\Lambda. \tag{109}$$

The solution reads

$$\mathrm{e}^{-2A} = \sigma - \frac{2M}{r^{\tilde{d}}} + \frac{Q^2}{r^{2\tilde{d}}} + \frac{\Lambda r^2}{\tilde{d}+2}, \tag{110}$$

where

$$Q^2 = \frac{b^2}{2\tilde{d}(D-2)}, \tag{111}$$

and M is the integration constant. For $\Lambda < 0$ and $\sigma = 0, -1$ this is the standard two-parametric family of charged topological black holes in D dimensions. Thus, by complete integration of the field equations we prove that this solution is unique within the class considered.

VII.   CONCLUSIONS

In this paper, we have constructed the general solution to the metric-dilaton-antisymmetric form action in arbitrary dimensions describing black p-branes with homogeneous transverse spaces of different topologies. The solution contains eight free parameters subject to a constraint involving four of them. When the asymptotic flatness condition is imposed on solutions with spherical transverse space, the number of independent parameters is reduced by two. The requirement of the existence of a regular horizon further reduces this number by two or three depending on whether the horizon is non-degenerate or not. Finally we are left with the standard solutions which are determined by their mass and charge densities and do not contain any extra parameters (apart from the asymptotic value of the dilaton). Our result proves the uniqueness of the standard asymptotically flat p-brane solutions with spherical transverse space for both non-degenerate and degenerate event horizons. (In particular, we extend the uniqueness proof for the multidimensional static dilatonic black holes [13] in the case of a degenerate horizon, assuming spherical symmetry). The only physical parameters of a singly charged p-brane are its mass and charge densities and the asymptotic value of the dilaton. All extra parameters arising in the course of complete integration of the field equations are removed by requiring asymptotic flatness and absence of naked singularities. The p-brane solutions with extra parameters previously suggested in the literature (the three-parameter solution of [6], the four-parameter solution of [14], the solutions discussed in [20, 21, 22, 23]) therefore are not free of naked singularities, unless the extra parameters are given some particular values.

We have also obtained the complete solutions for p-branes with hyperbolic or toroidal transverse spaces, but these were found to be incompatible with the requirement of asymptotic flatness. In fact, this is what could be expected, since in the better studied case of black holes the topological censorship theorem forbids asymptotically flat solutions with non-trivial topology of the horizon. Our considerations prove topological censorship for p-branes, assuming homogeneity of the transverse space, and we believe that this conjecture is true under less restrictive assumptions as well.

Finally we attempted to find general black brane solutions in the presence of a cosmological constant, but the system of equations is not fully integrable in this case. We were able to find the complete solution only in the case $d = 1$, i.e. for the black hole. For a negative cosmological constant the solutions exist for all three topologies of the event horizon and coincide with the multidimensional topological black holes previously presented in [38]. Thus our complete integration procedure proves the uniqueness of charged topological black holes in arbitrary dimensions under the assumption of homogeneity of the transverse space.

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